

NONCOMMUTATIVE BUNDLES OVER THE MULTI-PULLBACK QUANTUM COMPLEX PROJECTIVE PLANE

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ABSTRACT. We equip the multi-pullback C^* -algebra $C(S_H^5)$ of a noncommutative-deformation of the 5-sphere with a free $U(1)$ -action, and show that its fixed-point subalgebra is isomorphic with the C^* -algebra of the multi-pullback quantum complex projective plane. Our main result is the stable non-triviality of the dual tautological line bundle associated to the action. We prove it by combining Chern-Galois theory with the Milnor connecting homomorphism in K -theory. Using the Mayer-Vietoris six-term exact sequences and the functoriality of the Künneth formula, we also compute the K -groups of $C(S_H^5)$.

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INTRODUCTION

This paper is a part of a bigger project devoted to the K -theory of multi-pullback noncommutative deformations of free actions on spheres defining complex and real projective spaces. The lowest-dimensional complex case is worked out in [BHMS05, HMS06b] with the help of index theory. Herein we focus on the triple-pullback quantum complex projective plane $\mathbb{C}P_{\mathcal{T}}^2$ [HKZ12] and its quantum 5-sphere S_H^5 . Upgrading from pullback C^* -algebras of [BHMS05, HMS06b] to triple-pullback C^* -algebras requires a significant change of methods. In particular, we have to take care of the cocycle condition, as explained in Section 1.2.3, to compute the K -groups of $C(S_H^5)$ and $C(\mathbb{C}P_{\mathcal{T}}^2)$ in Section 3 and [R-J12] respectively.

The main theorem of the paper is:

Theorem 2.4 The section module $C(S_H^5)_u$ of the dual tautological line bundle over $\mathbb{C}P_{\mathcal{T}}^2$ is *not stably free* as a left $C(\mathbb{C}P_{\mathcal{T}}^2)$ -module.

The result is derived by comparing two idempotents: one coming from Chern-Galois theory applied to the $U(1)$ -action on $C(S_H^5)$, and the other one obtained by applying a formula (1.12) for the Milnor connecting homomorphism in a K -theory exact sequence. It is the same strategy that was used to determine non-trivial generators of the K_0 -group of Heegaard quantum lens spaces [HRZ13].

To explain a wider background and make the paper self-contained, we begin with a review of basic building blocks that are subsequently assembled into new results. Concerning notation, we use the unadorned tensor product \otimes to denote the minimal (spatial) tensor product of C^* -algebras and \otimes_{alg} to denote the algebraic tensor product.

1. PRELIMINARIES

1.1. From the Toeplitz algebra to quantum projective spaces.

1.1.1. *Toeplitz algebra.* There are different ways to introduce the Toeplitz algebra \mathcal{T} . Herein we define it as the universal C^* -algebra generated by one isometry s , i.e. an element satisfying the relation $s^*s = 1$. (Throughout the paper s will always mean the generating isometry of \mathcal{T} .) Likewise, u will always mean the unitary element generating the C^* -algebra $C(S^1)$ of all continuous complex-valued functions on the unit circle $S^1 := \{x \in \mathbb{C} \mid |x| = 1\}$. By mapping s to u , we obtain the well-known short exact sequence of C^* -algebras [C-LA I, C-LA II]:

$$(1.1) \quad 0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \xrightarrow{\sigma} C(S^1) \longrightarrow 0.$$

We consider the Toeplitz algebra as the C^* -algebra of continuous functions on a *quantum disc*. To justify this point of view, we take the family of universal C^* -algebras generated by x satisfying $x^*x - qxx^* = 1 - q$, $\|x\| = 1$, $q \in [0, 1]$ [KL93]. For $q \neq 1$, the norm condition is implied by the relation, and can be omitted. For $q = 1$, it yields precisely the C^* -algebra $C(D)$ of all continuous complex-valued functions on the unit disc $D := \{x \in \mathbb{C} \mid |x| \leq 1\}$. Finally, for $q = 0$, we get the Toeplitz algebra. Thus we obtain the \mathcal{T} as a q -deformation of $C(D)$.

Both the Toeplitz algebra \mathcal{T} and $C(S^1)$ are examples of graph C^* -algebras [FLR00]. Graph C^* -algebras are generated by partial isometries. They come naturally equipped with a $U(1)$ -action given by rephasing these partial isometries by unitary complex numbers. This $U(1)$ -action is called the gauge action. A key feature of the symbol map σ is that it is equivariant with respect to the gauge actions.

1.1.2. *Projective spaces.* Projective spaces of dimension $n \in \mathbb{N}$ over a topological field \mathbb{K} are defined as follows:

$$(1.2) \quad \mathbb{K}P^n := \{(x_0, \dots, x_n) \in \mathbb{K}^{n+1} \setminus (0, \dots, 0)\} / \sim, \\ (x_0, \dots, x_n) \sim (y_0, \dots, y_n) \iff \exists \lambda \in \mathbb{K} \setminus \{0\} : (x_0, \dots, x_n) = \lambda(y_0, \dots, y_n).$$

We denote the equivalence class of (x_0, \dots, x_n) by $[x_0 : \dots : x_n]$. There is the canonical affine open covering of the thus defined projective spaces:

$$(1.3) \quad \forall i \in \{0, \dots, n\} : U_i := \{[x_0 : \dots : x_n] \in \mathbb{K}P^n \mid x_i \neq 0\} \xrightarrow{\widetilde{\psi}_i} \mathbb{K}^n.$$

The above homeomorphisms are given by

$$(1.4) \quad \widetilde{\psi}_i([x_0 : \dots : x_n]) := \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i} \right).$$

Let us now focus our attention on $\mathbb{K} = \mathbb{C}$. To express the covering subsets in C^* -algebraic terms, we choose closed rather than open coverings. To this end, we define the following closed refinement of the affine covering:

$$(1.5) \quad \forall i \in \{0, \dots, n\} : V_i := \{[x_0 : \dots : x_n] \in \mathbb{C}P^n \mid |x_i| = \max\{|x_0|, \dots, |x_n|\}\} \cong D^n.$$

Here the homeomorphisms are given by appropriate restrictions of $\widetilde{\psi}_i$'s denoted by ψ_i . We use the covering $\{V_i\}_i$ to present $\mathbb{C}P^n$ as a multi-pushout. More precisely, we pick indices $0 \leq i < j \leq n$, denote by ψ_{ij} the restriction of ψ_i to $V_i \cap V_j$, and take the following commutative diagram:

$$(1.6) \quad \begin{array}{ccccc} & & \mathbb{C}P^n & & \\ & \swarrow \psi_i & \nearrow \psi_j & \searrow \psi_j & \swarrow \psi_i \\ D^n & \xleftarrow{\psi_i} & V_i & & V_j \xrightarrow{\psi_j} D^n \\ \uparrow & & \swarrow \psi_{ij} & & \searrow \psi_{ji} \\ D^{j-1} \times S^1 \times D^{n-j} & \xleftarrow{\psi_{ij}} & V_i \cap V_j & \xrightarrow{\psi_{ji}} & D^i \times S^1 \times D^{n-i-1}. \end{array}$$

1.1.3. *Quantum complex projective spaces.* Now we combine the foregoing presentation of projective spaces with the idea that the Toeplitz algebra is the C^* -algebra of functions on a quantum unit disc to construct a new type of quantum projective spaces [HKZ12]. To define them, first we excise from diagram (1.6) its middle square, and dualise it to the multi-pullback diagram of

unital commutative C^* -algebras of functions on appropriate compact Hausdorff spaces:

$$(1.7) \quad \begin{array}{ccccc} & & C(\mathbb{C}P^n) & & \\ & \swarrow \text{dashed} & & \searrow \text{dashed} & \\ C(D)^{\otimes n} & & & & C(D)^{\otimes n} \\ \downarrow \pi_j^i & \swarrow \pi_i^j & & \searrow & \downarrow \\ C(D)^{\otimes j-1} \otimes C(S^1) \otimes C(D)^{\otimes n-j} & \xleftarrow{(\psi_{ji} \circ \psi_{ij}^{-1})^*} & C(D)^{\otimes i} \otimes C(S^1) \otimes C(D)^{\otimes n-i-1} \end{array}$$

This yields a multi-pullback presentation of $C(\mathbb{C}P^n)$. Then we leave $C(S^1)$ unchanged and replace $C(D)$ by \mathcal{T} . It turns out that the formulae for $(\psi_{ji} \circ \psi_{ij}^{-1})^*$'s, π_j^i 's and π_i^j 's continue to make sense after these replacements, so that quantum projective spaces can be defined as Pedersen's *multi-pullback C^* -algebras* (see [P-GK99, CM00])

$$(1.8) \quad B^\pi := \left\{ (b_i)_i \in \prod_{i \in J} B_i \mid \pi_j^i(b_i) = \pi_i^j(b_j), \forall i, j \in J, i \neq j \right\},$$

where $\{\pi_j^i : B_i \rightarrow B_{ij} = B_{ji}\}_{i,j \in J, i \neq j}$ is the family of C^* -homomorphisms defined through commutative diagram (1.7) with $C(D)$ replaced by \mathcal{T} .

1.2. Mayer-Vietoris six-term exact sequence. For a one-surjective pullback diagram of C^* -algebras

$$(1.9) \quad \begin{array}{ccc} & A & \\ \swarrow & & \searrow \\ B_0 & & B_1 \\ \searrow \pi_0 & & \swarrow \pi_1 \\ & B_{01} & \end{array}$$

there exists the Mayer-Vietoris six-term exact sequence (e.g., see [B-B98, Theorem 21.2.2] [BHMS05, Section 1.3], [S-C84]):

$$(1.10) \quad \begin{array}{ccccc} K_0(A) & \longrightarrow & K_0(B_0 \oplus B_1) & \longrightarrow & K_0(B_{01}) \\ \partial_{10} \uparrow & & & & \downarrow \partial_{01} \\ K_1(B_{01}) & \longleftarrow & K_1(B_0 \oplus B_1) & \longleftarrow & K_1(A). \end{array}$$

In our applications of this exact sequence, we will need explicit formulae for connecting homomorphisms ∂_{10} and ∂_{01} .

1.2.1. Odd-to-even connecting homomorphism. Following the celebrated Milnor's construction of an odd-to-even connecting homomorphism in algebraic K -theory [M-J71], one can derive an explicit formula for this homomorphism [R-A, DHHMW12], and adapt it to unital C^* -algebras (see [HRZ13, Section 0.4] for an argument of Nigel Higson).

Theorem 1.1. *Let $U \in GL_n(B_{01})$, $(\text{id} \otimes \pi_0)(c) = U^{-1}$ and $(\text{id} \otimes \pi_0)(d) = U$. Denote by I_n the identity matrix of size n , and put*

$$(1.11) \quad p_U := \begin{pmatrix} (c(2-dc)d, 1) & (c(2-dc)(1-dc), 0) \\ ((1-dc)d, 0) & ((1-dc)^2, 0) \end{pmatrix} \in M_{2n}(A).$$

Then p_U is an idempotent and the formula

$$(1.12) \quad \partial_{10}([U]) := [p_U] - [I_n]$$

defines an odd-to-even connecting homomorphism $\partial_{10} : K_1(B_{01}) \rightarrow K_0(A)$ in the Mayer-Vietoris six-term exact sequence (1.10).

1.2.2. *Even-to-odd connecting homomorphism.* Combining [BM, Theorem 1.18] with [B-B98, Section 9.3.2], we obtain:

Theorem 1.2. *Let $p \in M_n(B_{01})$ be a projection, $(\text{id} \otimes \pi_0)(Q_p) = p$, $Q_p^* = Q_p$, and I_n be the identity matrix of size n . Then the formula*

$$(1.13) \quad \partial_{10}([p]) := [(e^{2\pi i Q_p}, I_n)]$$

defines an even-to-odd connecting homomorphism in the Mayer-Vietoris six-term exact sequence (1.10).

1.2.3. *Cocycle condition for multi-pullback C^* -algebras.* We construct algebras of functions on quantum spaces as multi-pullbacks of C^* -algebras. To make sure that this construction dually corresponds to the presentation of a quantum space as a “union of closed subspaces” (no self gluings of closed subspaces or their partial multi-pushouts; see [HZ12] for an in-depth discussion of these issues), we assume the cocycle condition. It allows us to apply the Mayer-Vietoris six-term exact sequence to multi-pullback C^* -algebras by guaranteeing surjectivity of appropriate $*$ -homomorphisms.

First we need some auxilliary definitions. Let $(\pi_j^i : A_i \rightarrow A_{ij})_{i,j \in J, i \neq j}$ be a finite family of surjective C^* -algebra homomorphisms. For all distinct $i, j, k \in J$, we define $A_{jk}^i := A_i / (\ker \pi_j^i + \ker \pi_k^i)$ and denote by $[\cdot]_{jk}^i : A_i \rightarrow A_{jk}^i$ the canonical surjections. Next, we introduce the family of maps

$$(1.14) \quad \pi_k^{ij} : A_{jk}^i \longrightarrow A_{ij} / \pi_j^i(\ker \pi_k^i), \quad [b_i]_{jk}^i \longmapsto \pi_j^i(b_i) + \pi_j^i(\ker \pi_k^i),$$

for all distinct $i, j, k \in J$. Note that they are isomorphisms when all π_j^i 's are surjective C^* -algebra homomorphisms, as assumed herein.

We say [CM00, in Proposition 9] that a finite family $(\pi_j^i : A_i \rightarrow A_{ij})_{i,j \in J, i \neq j}$ of C^* -algebra surjections satisfies the *cocycle condition* if and only if, for all distinct $i, j, k \in J$,

- (1) $\pi_j^i(\ker \pi_k^i) = \pi_i^j(\ker \pi_k^j)$,
- (2) the isomorphisms $\varphi_k^{ij} := (\pi_k^{ij})^{-1} \circ \pi_k^{ji} : A_{ik}^j \rightarrow A_{jk}^i$ satisfy $\varphi_j^{ik} = \varphi_k^{ij} \circ \varphi_i^{jk}$.

One proves ([HZ12, Theorem 1]) that a finite family $(\pi_j^i : A_i \rightarrow A_{ij})_{i,j \in J, i \neq j}$ of C^* -algebra surjections satisfies the cocycle condition if and only if, for all $K \subsetneq J$, $k \in J \setminus K$, and $(b_i)_{i \in K} \in \bigoplus_{i \in K} A_i$ such that $\pi_j^i(b_i) = \pi_i^j(b_j)$ for all distinct $i, j \in K$, there exists $b_k \in A_k$ such that also $\pi_k^i(b_i) = \pi_i^k(b_k)$ for all $i \in K$. One can easily see that dually this corresponds to the statement “a quantum space is a pushout of parts, and all partial pushouts are embedded in this quantum space”. This is what we usually have in mind when constructing a space from parts.

1.3. Actions of compact Hausdorff groups on unital C^* -algebras. To use the language of strong connections [H-PM96] and facilitate some computations, we need to transform actions of compact Hausdorff groups on unital C^* -algebras to coactions of their C^* -algebras on unital C^* -algebras. More precisely, let A be a unital C^* -algebra and G a compact Hausdorff group with a group homomorphism $\alpha: G \ni g \mapsto \alpha_g \in \text{Aut}(A)$. Then

$$(1.15) \quad \delta_\alpha: A \longrightarrow C(G, A) \cong A \otimes C(G), \quad \delta_\alpha(a)(g) := \alpha_g(a).$$

We will use the thus related action and coaction interchangeably.

Furthermore, for any compact Hausdorff group G , we can define the Hopf-algebraic structure on $C(G)$ due to its commutativity:

- the comultiplication $\Delta: C(G) \rightarrow C(G) \otimes C(G)$,
- the counit $\varepsilon: C(G) \rightarrow \mathbb{C}$,
- and the antipode $S: C(G) \rightarrow C(G)$

are respectively the pullbacks of the group multiplication, the embedding of the neutral element into G , and the inverting map $G \ni g \mapsto g^{-1} \in G$. We can also use the Heynemann-Sweedler notation (with the summation sign suppressed) for coactions and comultiplications:

- $\delta(a) =: a_{(0)} \otimes a_{(1)}, \quad \delta(a)(g) = (a_{(0)} \otimes a_{(1)})(g) = a_{(0)}a_{(1)}(g),$
- $\Delta(h) =: h_{(1)} \otimes h_{(2)}, \quad \Delta(h)(g_1, g_2) = (h_{(1)} \otimes h_{(2)})(g_1, g_2) = h_{(1)}(g_1)h_{(2)}(g_2) = h(g_1g_2).$

In particular, for $G = U(1)$, the antipode is determined by $S(u) = u^{-1}$, the counit by $\varepsilon(u) = 1$, and finally the comultiplication by $\Delta(u) = u \otimes u$. The coaction of $C(U(1))$ on \mathcal{T} coming from the aforementioned (Section 1.1.1) gauge action of $U(1)$ on \mathcal{T} becomes

$$(1.16) \quad \delta: \mathcal{T} \longrightarrow \mathcal{T} \otimes C(U(1)), \quad \delta(s) := s \otimes u.$$

1.3.1. Freeness. Following [E-DA00], we say that an action of a compact Hausdorff group G on a unital C^* -algebra A is *free* if and only if the induced coaction satisfies the following norm-density condition:

$$(1.17) \quad \{(x \otimes 1)\delta(y) \mid x, y \in A\}^{\text{cls}} = A \otimes C(G).$$

Here “cls” stands for “closed linear span”.

Next, let us denote by $\mathcal{O}(G)$ the dense Hopf $*$ -subalgebra spanned by the matrix coefficients of finite-dimensional representations. We define the *Peter-Weyl subalgebra* of A as

$$(1.18) \quad \mathcal{P}_G(A) := \left\{ a \in A \mid \delta(a) \in A \otimes_{\text{alg}} \mathcal{O}(G) \right\}.$$

One shows that it is an $\mathcal{O}(G)$ -comodule algebra which is a dense $*$ -subalgebra of A . (See [S-PM11] and references therein.) Moreover, the C^* -algebraic freeness condition on a G - C^* -algebra A is equivalent to the algebraic *principality* condition on the $\mathcal{O}(G)$ -comodule algebra $\mathcal{P}_G(A)$ [BDH]. This allows us to use crucial algebraic tools without leaving the ground of C^* -algebras.

1.3.2. *Strong connections and principal comodule algebras.* One can prove (see [BH] and references therein) that a comodule algebra is principal if and only if it admits a strong connection. Therefore we will treat the existence of a strong connection as a condition defining the principality of a comodule algebra and avoid the original definition of a principal comodule algebra. The latter is important when going beyond coactions that are algebra homomorphisms — then the existence of a strong connection is implied by principality but we do not have the reverse implication [BH04].

Let G be a compact Hausdorff group acting on a unital C^* -algebra A . A *strong connection* ℓ on A is a unital linear map $\ell : \mathcal{O}(G) \rightarrow \mathcal{P}_G(A) \otimes_{\text{alg}} \mathcal{P}_G(A)$ satisfying:

- (1) $(\text{id} \otimes \delta) \circ \ell = (\ell \otimes \text{id}) \circ \Delta$, $((S \otimes \text{id}) \circ \text{flip} \circ \delta) \otimes \text{id} \circ \ell = (\text{id} \otimes \ell) \circ \Delta$;
- (2) $m \circ \ell = \varepsilon$, where $m : \mathcal{P}_G(A) \otimes_{\text{alg}} \mathcal{P}_G(A) \rightarrow \mathcal{P}_G(A)$ is the multiplication map.

Here we abuse notation by using the same symbol for a restriction-corestriction of a map as for the map itself.

1.3.3. *Associated projective modules.* Let $\varrho : G \rightarrow GL(V)$ be a representation of a compact Hausdorff group G on a complex vector space V , and $\alpha : G \rightarrow \text{Aut}(A)$ be an action on a unital C^* -algebra A . Then the *associated module* $\mathcal{P}_G(A) \square^{\varrho} V$ is, by definition,

$$(1.19) \quad \left\{ x \in \mathcal{P}_G(A) \otimes_{\text{alg}} V \mid \forall g \in G : (\alpha_g \otimes \text{id})(x) = (\text{id} \otimes \varrho(g^{-1}))(x) \right\}.$$

It is a left module over the fixed-point subalgebra $A^{\alpha} := \{a \in A \mid \forall g \in G : \alpha_g(a) = a\} =: A^{U(1)}$.

If V is finite dimensional and α is free, then $\mathcal{P}_G(A) \square^{\varrho} V$ is finitely generated projective [HM99]. We think of it as the section module of an *associated noncommutative vector bundle*. Furthermore, if $\dim V = 1$ and $\gamma : G \rightarrow GL(\mathbb{C})$ is a representation, then we obtain:

$$(1.20) \quad \mathcal{P}_G(A) \square^{\gamma} \mathbb{C} = \{a \in A \mid \delta(a) = a \otimes S(\gamma)\} =: A_{\gamma^{-1}}.$$

Modules A_{γ} are called *spectral subspaces*. We think of them as the section modules of associated noncommutative *line* bundles.

Now it is quite easy to apply Chern-Galois theory [BH04, Theorem 3.1], and compute an idempotent E_{γ} representing the associated module A_{γ} using a strong connection ℓ :

$$(1.21) \quad A_{\gamma} \cong (A^{\alpha})^n E^{\gamma}, \quad E_{ij}^{\gamma} := \gamma_i^R \gamma_j^L, \quad \ell(\gamma) =: \sum_{k=1}^n \gamma_k^L \otimes \gamma_k^R \in A_{\gamma^{-1}} \otimes_{\text{alg}} A_{\gamma},$$

where $\{\gamma_k^L\}_k$ is a linearly independent set.

1.3.4. *Gauging coactions.* Consider $A \otimes C(G)$ as a C^* -algebra with the diagonal coaction

$$(1.22) \quad p \otimes h \longmapsto p_{(0)} \otimes h_{(1)} \otimes p_{(1)} h_{(2)},$$

and denote by $(A \otimes C(G))_R$ the same C^* -algebra but now equipped with the coaction on the rightmost factor

$$(1.23) \quad p \otimes h \longmapsto p \otimes h_{(1)} \otimes h_{(2)}.$$

Then the following map is a G -equivariant (i.e., intertwining the coactions) *gauge* isomorphism of C^* -algebras:

$$(1.24) \quad \widehat{\kappa} : (A \otimes C(G)) \longrightarrow (A \otimes C(G))_R, \quad a \otimes h \longmapsto a_{(0)} \otimes a_{(1)}h.$$

Its inverse is explicitly given by

$$(1.25) \quad \widehat{\kappa}^{-1} : (A \otimes H)_R \longrightarrow (A \otimes H), \quad a \otimes h \longmapsto a_{(0)} \otimes S(a_{(1)})h.$$

2. DUAL TAUTOLOGICAL LINE BUNDLE

2.1. Quantum complex projective plane. We consider the case $n = 2$ of the multi-Toeplitz deformations [HKMZ11, Section 2] of the complex projective spaces. The C^* -algebra of our quantum projective plane is given as the triple-pullback of the following diagram:

$$(2.1) \quad \begin{array}{ccccc} \mathcal{T} \otimes \mathcal{T} & & \mathcal{T} \otimes \mathcal{T} & & \mathcal{T} \otimes \mathcal{T} \\ \searrow \sigma_1 & \swarrow \Psi_{01} \circ \sigma_1 & \searrow \sigma_2 & \swarrow \Psi_{12} \circ \sigma_2 & \\ C(S^1) \otimes \mathcal{T} & & \mathcal{T} \otimes C(S^1) & & \\ \searrow \sigma_2 & & \swarrow \Psi_{02} \circ \sigma_1 & & \\ & \mathcal{T} \otimes C(S^1) & & & \end{array}$$

Here $\sigma_1 := \sigma \otimes \text{id}$, $\sigma_2 := \text{id} \otimes \sigma$, and

$$(2.2) \quad \begin{aligned} C(S^1) \otimes \mathcal{T} \ni v \otimes t &\xrightarrow{\Psi_{01}} S(t_{(1)}v) \otimes t_{(0)} \in C(S^1) \otimes \mathcal{T}, \\ C(S^1) \otimes \mathcal{T} \ni v \otimes t &\xrightarrow{\Psi_{02}} t_{(0)} \otimes S(t_{(1)}v) \in \mathcal{T} \otimes C(S^1), \\ \mathcal{T} \otimes C(S^1) \ni t \otimes v &\xrightarrow{\Psi_{12}} t_{(0)} \otimes S(t_{(1)}v) \in \mathcal{T} \otimes C(S^1), \end{aligned}$$

where $\mathcal{T} \ni t \mapsto t_{(0)} \otimes t_{(1)} \in \mathcal{T} \otimes C(S^1)$ is the coaction of (1.16).

2.2. Quantum complex projective plane $\mathbb{C}P^2_\mathcal{T}$ as quotient space $S^5_H/U(1)$. Consider the following triple-pullback diagram in which every homomorphism is given by the symbol map on the appropriate factor and identity otherwise:

$$(2.3) \quad \begin{array}{ccccc} C(S^1) \otimes \mathcal{T} \otimes \mathcal{T} & & \mathcal{T} \otimes C(S^1) \otimes \mathcal{T} & & \mathcal{T} \otimes \mathcal{T} \otimes C(S^1) \\ \searrow & \swarrow & \searrow & \swarrow & \\ C(S^1) \otimes C(S^1) \otimes \mathcal{T} & & \mathcal{T} \otimes C(S^1) \otimes C(S^1) & & \\ \searrow & & \swarrow & & \\ & C(S^1) \otimes \mathcal{T} \otimes C(S^1) & & & \end{array}$$

Definition 2.1. The multi-pullback C^* -algebra of the family of C^* -epimorphisms in (2.3) is called the C^* -algebra of the Heegaard odd quantum sphere S^5_H and denoted $C(S^5_H)$.

Using the coaction (1.16) on \mathcal{T} and the comultiplication on $C(S^1) = C(U(1))$, we define the diagonal coaction on each C^* -algebra of the above diagram as in Section 1.3.4. The diagram is evidently equivariant with respect to this coaction because the symbol map is equivariant. Therefore $C(S_H^5)$ is a $U(1)$ - C^* -algebra. We call this $U(1)$ -action on $C(S_H^5)$ *diagonal*.

In order to compute the fixed-point subalgebra for the above diagonal $U(1)$ -action, we need to gauge it to an action on tensor products that acts on the rightmost $C(S^1)$ -factor alone. Our goal is to show that the fixed-point subalgebra is isomorphic with $C(\mathbb{C}P_7^2)$. To this end, we double the three targets of all homomorphisms in (2.3) to three pairs of sibling targets, so that

$$(2.4) \quad \begin{array}{ccc} C(S^1) \otimes \mathcal{T} \otimes \mathcal{T} & & \mathcal{T} \otimes C(S^1) \otimes \mathcal{T} \\ & \searrow & \swarrow \\ & C(S^1) \otimes C(S^1) \otimes \mathcal{T} & \end{array}$$

becomes

$$(2.5) \quad \begin{array}{ccc} C(S^1) \otimes \mathcal{T} \otimes \mathcal{T} & & \mathcal{T} \otimes C(S^1) \otimes \mathcal{T} \\ \downarrow & & \downarrow \\ C(S^1) \otimes C(S^1) \otimes \mathcal{T} & \xleftarrow{\text{id}} & C(S^1) \otimes C(S^1) \otimes \mathcal{T}, \end{array}$$

and other subdiagrams are transformed in the same fashion. Then we permute the factors in the tensor products in the top row to make $C(S^1)$ always the rightmost factor, and permute the target tensor products accordingly:

$$(2.6) \quad \begin{array}{ccc} \mathcal{T} \otimes \mathcal{T} \otimes C(S^1) & & \mathcal{T} \otimes \mathcal{T} \otimes C(S^1) \\ \sigma \otimes \text{id} \otimes \text{id} \downarrow & & \downarrow \sigma \otimes \text{id} \otimes \text{id} \\ C(S^1) \otimes \mathcal{T} \otimes C(S^1) & \xleftarrow{T_{13}} & C(S^1) \otimes \mathcal{T} \otimes C(S^1). \end{array}$$

Here the horizontal arrow is just the flip of the outer factors. Again, we apply analogous procedures to the other two subdiagrams. Due to the commutativity of $C(S^1)$, the thus obtained triple-pullback diagram is equivariant for the diagonal coaction, and the C^* -algebra it defines is equivariantly isomorphic with the multi-pullback C^* -algebra defined by diagram (2.3).

Now we are ready to gauge the diagonal action as explained in Section 1.3.4. Conjugating $T_{13} \circ (\sigma \otimes \text{id} \otimes \text{id})$ by the gauge isomorphisms (1.24)–(1.25), using the commutativity and cocommutativity of $C(S^1) = C(U(1))$, along the lines of [HKMZ11, Section 5.2], we get:

$$(2.7) \quad \begin{aligned} & (\tilde{g} \circ T_{13} \circ (\sigma \otimes \text{id} \otimes \text{id}) \circ g^{-1})(r \otimes t \otimes w) \\ &= (\tilde{g} \circ T_{13} \circ (\sigma \otimes \text{id} \otimes \text{id}))(r_{(0)} \otimes t_{(0)} \otimes S(r_{(1)}t_{(1)})w) \\ &= (\tilde{g} \circ T_{13})(\sigma(r)_{(1)} \otimes t_{(0)} \otimes S(\sigma(r)_{(2)}t_{(1)})w) \\ &= \tilde{g}(S(\sigma(r)_{(2)}t_{(1)})w \otimes t_{(0)} \otimes \sigma(r)_{(1)}) \\ &= S(\sigma(r)_{(3)}t_{(3)})w_{(1)} \otimes t_{(0)} \otimes S(\sigma(r)_{(2)}t_{(2)})w_{(2)}t_{(1)}\sigma(r)_{(1)} \\ &= S(\sigma(r)t_{(1)})w_{(1)} \otimes t_{(0)} \otimes w_{(2)}. \end{aligned}$$

Here g and \tilde{g} are the gauge isomorphisms on $(\mathcal{T} \otimes \mathcal{T}) \otimes C(U(1))$ and $(C(S^1) \otimes \mathcal{T}) \otimes C(U(1))$ respectively.

Much in the same way, we treat the remaining two subdiagrams of diagram (2.3). Summarizing, for $0 \leq i < j \leq 2$, the permuted and then gauged subdiagrams become:

$$(2.8) \quad \begin{array}{ccc} \mathcal{T}^{\otimes 2} \otimes C(S^1) & & \mathcal{T}^{\otimes 2} \otimes C(S^1) \\ \sigma_j \downarrow & & \sigma_{i+1} \downarrow \\ \mathcal{T}^{\otimes j-1} \otimes C(S^1) \otimes \mathcal{T}^{\otimes 2-j} \otimes C(S^1) & \xleftarrow{\Psi_{ij}^S} & \mathcal{T}^{\otimes i} \otimes C(S^1) \otimes \mathcal{T}^{\otimes 1-i} \otimes C(S^1), \end{array}$$

where

$$(2.9) \quad \begin{aligned} \Psi_{01}^S(v \otimes t \otimes w) &:= S(vt_{(1)})w_{(1)} \otimes t_{(0)} \otimes w_{(2)}, \\ \Psi_{02}^S(v \otimes t \otimes w) &:= t_{(0)} \otimes S(vt_{(1)})w_{(1)} \otimes w_{(2)}, \\ \Psi_{12}^S(t \otimes v \otimes w) &:= t_{(0)} \otimes S(t_{(1)}v)w_{(1)} \otimes w_{(2)}. \end{aligned}$$

The triple-pullback C^* -algebra of the family (2.8) is denoted by $C(S_H^5)_R$. It is a $U(1)$ - C^* -algebra that is equivariantly isomorphic with $C(S_H^5)$:

$$(2.10) \quad \begin{aligned} C(S_H^5) \ni (v^0 \otimes t^0 \otimes r^0, t^1 \otimes v^1 \otimes r^1, t^2 \otimes r^2 \otimes v^2) \longmapsto \\ (t^0_{(0)} \otimes r^0_{(0)} \otimes t^0_{(1)}r^0_{(1)}v^0, t^1_{(0)} \otimes r^1_{(0)} \otimes t^1_{(1)}r^1_{(1)}v^1, t^2_{(0)} \otimes r^2_{(0)} \otimes t^2_{(1)}r^2_{(1)}v^2) \in C(S_H^5)_R. \end{aligned}$$

This isomorphism yields an isomorphism of fixed-point subalgebras $C(S_H^5)^{U(1)} \cong C(S_H^5)_R^{U(1)}$. Since the $U(1)$ -action in the triple-pullback diagram defining $C(S_H^5)_R$ acts only on the rightmost factor, we conclude that $C(S_H^5)_R^{U(1)}$ is the triple-pullback C^* -algebra obtained by removing all rightmost factors in (2.8) and taking $w = 1$ in (2.9). Finally, since the isomorphisms in (2.9) thus become the isomorphisms in (2.2), so that (2.8) becomes the defining triple-pullback diagram (2.1) of $C(\mathbb{C}P_{\mathcal{T}}^2)$, we infer that $C(S_H^5)^{U(1)} \cong C(\mathbb{C}P_{\mathcal{T}}^2)$.

2.3. Strong connection for the diagonal $U(1)$ -action on $C(S_H^5)$.

Theorem 2.2. *The diagonal $U(1)$ -action on $C(S_H^5)$ is free.*

Proof. We prove the claim by constructing a strong connection on the Peter-Weyl comodule algebra $\mathcal{P}_{U(1)}(C(S_H^5))$ for the diagonal coaction $\delta: C(S_H^5) \rightarrow C(S_H^5) \otimes C(U(1))$. Let u be the generating unitary of $C(S^1)$ and s be the generating isometry of \mathcal{T} . Consider the following isometries in $C(S_H^5)$:

$$(2.11) \quad \begin{aligned} a &:= (u \otimes 1 \otimes 1, s \otimes 1 \otimes 1, s \otimes 1 \otimes 1), \\ b &:= (1 \otimes s \otimes 1, 1 \otimes u \otimes 1, 1 \otimes s \otimes 1), \\ c &:= (1 \otimes 1 \otimes s, 1 \otimes 1 \otimes s, 1 \otimes 1 \otimes u). \end{aligned}$$

They all commute and satisfy the equation:

$$(2.12) \quad (1 - aa^*)(1 - bb^*)(1 - cc^*) = 0.$$

Now one can easily check that a strong connection

$$(2.13) \quad \ell: \mathcal{O}(U(1)) \longrightarrow \mathcal{P}_{U(1)}(C(S_H^5)) \otimes_{\text{alg}} \mathcal{P}_{U(1)}(C(S_H^5)) \subseteq C(S_H^5) \otimes C(S_H^5)$$

can be defined by the formulae:

$$(2.14) \quad \ell(1) = 1 \otimes 1, \quad \ell(u) = b^* \otimes b,$$

$$(2.15) \quad \ell(u^*) = a \otimes a^* + b \otimes b^* + c \otimes c^* - a \otimes a^* b b^* - a \otimes a^* c c^* - b \otimes b^* c c^* + a \otimes a^* b b^* c c^*.$$

Indeed, exactly as in [HMS06a, (4.6)], we can show inductively that the formula

$$(2.16) \quad \ell(x^n) := \ell(x)^{\langle 1 \rangle} \ell(x^{n-1}) \ell(x)^{\langle 2 \rangle}, \quad \ell(x) =: \ell(x)^{\langle 1 \rangle} \otimes \ell(x)^{\langle 2 \rangle} \text{ (summation suppressed),}$$

has the desired properties for x being any of the grouplikes u and u^* . \square

2.4. Stable non-freeness.

Definition 2.3. Let u be the generating unitary of $C(U(1))$ and $C(S_H^5) \xrightarrow{\delta} C(S_H^5) \otimes C(U(1))$ the diagonal coaction. We call the associated module

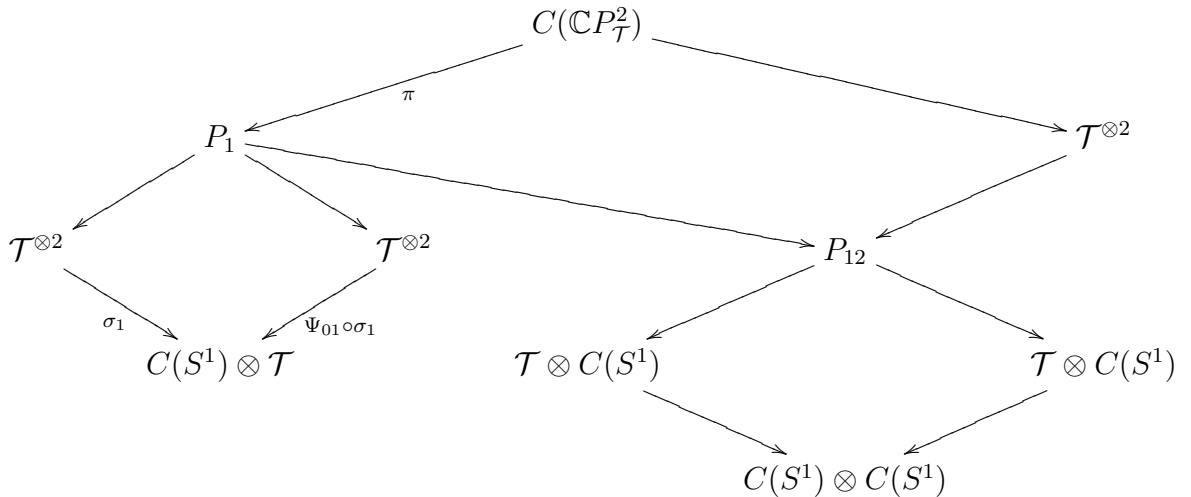
$$C(S_H^5)_u := \{x \in C(S_H^5) \mid \delta(x) = x \otimes u\}$$

the section module of the dual tautological line bundle over $\mathbb{C}P_{\mathcal{T}}^2$.

It follows from the existence of a strong connection on the Peter-Weyl comodule algebra $\mathcal{P}_{U(1)}(C(S_H^5))$ that $C(S_H^5)_u$ is a finitely generated projective module over $C(S_H^5)^{U(1)} \cong C(\mathbb{C}P_{\mathcal{T}}^2)$ [HM99]. Moreover, combining (1.21) with (2.14) proves that $C(S_H^5)_u$ is isomorphic as a left $C(S_H^5)^{U(1)}$ -module with $C(S_H^5)^{U(1)} b b^*$. This allows us to prove our main result:

Theorem 2.4. The section module $C(S_H^5)_u$ of the dual tautological line bundle over $\mathbb{C}P_{\mathcal{T}}^2$ is not stably free as a left $C(\mathbb{C}P_{\mathcal{T}}^2)$ -module.

Proof. The gauge isomorphism (2.10) turns the projection $b b^*$ representing the finitely generated projective module $C(S_H^5)_u$ to $(s s^* \otimes 1, 1 \otimes 1, 1 \otimes s s^*) \in C(\mathbb{C}P_{\mathcal{T}}^2)$. Plugging it into the iterated pullback diagram



and projecting via π to P_1 , we obtain $(s s^* \otimes 1, 1 \otimes 1)$.

Furthermore, consider the Mayer-Vietoris six-term exact sequence of the pullback diagram defining P_1 , and take unitary $u \otimes 1$ whose class generates $K_1(C(S^1) \otimes \mathcal{T})$. We know from the proof of [R-J12, Theorem 2.1] that $K_0(P_1) = \mathbb{Z} \oplus \mathbb{Z}$ with one \mathbb{Z} generated by $[1]$ and the other

\mathbb{Z} generated by $\partial_{10}([u \otimes 1])$. To compute the Milnor idempotent $p_{u \otimes 1}$ (see (1.11)), take a lifting of $u^{-1} \otimes 1$ to be $c := s^* \otimes 1$, and a lifting of $u \otimes 1$ to be $d := s \otimes 1$. Suppressing $\otimes 1$, we obtain

$$(2.17) \quad \partial_{10}([u \otimes 1]) = \left[\begin{pmatrix} (s^*(2 - ss^*)s, 1) & (s^*(2 - ss^*)(1 - ss^*), 0) \\ ((1 - ss^*)s, 0) & ((1 - ss^*)^2, 0) \end{pmatrix} \right] - [(1, 1)] = [(1 - ss^*, 0)].$$

Hence $[1] - \partial_{10}([u \otimes 1]) = [(ss^* \otimes 1, 1 \otimes 1)] = \pi_*[C(S_H^5)_u]$.

Finally, if $C(S_H^5)_u$ were stably free, then $\pi_*[C(S_H^5)_u] = n[1]$ for some $n \in \mathbb{N}$. This would contradict the just derived equality, so that $C(S_H^5)_u$ is not stably free. \square

3. K -GROUPS OF THE QUANTUM SPHERE S_H^5

We end this paper by showing that the K -groups of S_H^5 agree with their classical counterparts. Its C^* -algebra is the triple-pullback C^* -algebra (see 2.1), so that we can apply [R-J12, Corollary 1.5] to determine its K -theory.

3.1. Cocycle condition. The first step in applying [R-J12, Corollary 1.5] is verifying the cocycle condition (see Section 1.2.3).

Lemma 3.1. *The family (2.3) defining the triple-pullback C^* -algebra $C(S_H^5)$ satisfies the cocycle condition.*

Proof. It is straightforward to check the first part of the cocycle condition. We do it only in one case as all other cases are completely analogous. For $i = 2$, $j = 1$ and $k = 0$, we obtain:

$$(3.1) \quad \begin{aligned} \pi_1^2(\ker \pi_0^2) &= \pi_2^1(\ker \pi_0^1) \Leftrightarrow \\ \sigma_2(\ker \sigma_1) &= \sigma_3(\ker \sigma_1) \Leftrightarrow \\ \sigma_2(\mathcal{K} \otimes \mathcal{T} \otimes C(S^1)) &= \sigma_3(\mathcal{K} \otimes \mathcal{C}(S^1) \otimes \mathcal{T}) \Leftrightarrow \\ \mathcal{K} \otimes C(S^1) \otimes C(S^1) &= \mathcal{K} \otimes \mathcal{C}(S^1) \otimes C(S^1). \end{aligned}$$

For the second part we use the following notation

$$(3.2) \quad [\cdot]_{jk}^i : B_i \rightarrow B_i / (\ker \pi_j^i + \ker \pi_k^i), \quad [\cdot]_k^{ij} : B_{ij} \rightarrow B_{ij} / \pi_j^i(\ker \pi_k^i).$$

Again all cases are done in a similar way, so that we only check the case $i = 0$, $j = 1$, $k = 2$, i.e. we show that $\varphi_1^{02} = \varphi_2^{01} \circ \varphi_0^{12}$. For any $r \otimes t \otimes v \in \mathcal{T} \otimes \mathcal{T} \otimes C(S^1)$, the left hand side is:

$$(3.3) \quad \begin{aligned} \varphi_1^{02}([r \otimes t \otimes v]_{01}^2) &= ((\pi_1^{02})^{-1} \circ \pi_1^{20})([r \otimes t \otimes v]_{01}^2) \\ &= \left[((\pi_2^0)^{-1} \circ \pi_0^2)(r \otimes t \otimes v) \right]_{21}^0 \\ &= [(\sigma_3^{-1} \circ \sigma_1)(r \otimes t \otimes v)]_{21}^0 \\ &= [\sigma(r) \otimes t \otimes \omega(v)]_{21}^0, \end{aligned}$$

where ω is a linear splitting of σ . On the other hand, we obtain:

$$\begin{aligned}
 (\varphi_2^{01} \circ \varphi_0^{12})([r \otimes t \otimes v]_{01}^2) &= \varphi_2^{01} \left(\left[((\pi_2^1)^{-1} \circ \pi_1^2)(r \otimes t \otimes v) \right]_{20}^1 \right) \\
 &= \varphi_2^{01} \left([r \otimes \sigma(t) \otimes \omega(v)]_{02}^1 \right) \\
 &= ((\pi_2^{01})^{-1} \circ \pi_2^{10})([r \otimes \sigma(t) \otimes \omega(v)]_{02}^1) \\
 &= \left[((\pi_1^0)^{-1} \circ \pi_0^1)(r \otimes \sigma(t) \otimes \omega(v)) \right]_{12}^0 \\
 &= [\sigma(r) \otimes \omega(\sigma(t)) \otimes \omega(v)]_{12}^0 \\
 &= [\sigma(r) \otimes t \otimes \omega(v)]_{12}^0.
 \end{aligned}
 \tag{3.4}$$

Hence the left and the right hand side agree because $[\]_{jk}^i = [\]_{kj}^i$ for any set of distinct indices. \square

3.2. K -groups. We are now ready for:

Theorem 3.2. *The K -groups of the Heegaard quantum 5-sphere are:*

$$K_0(C(S_H^5)) = \mathbb{Z} = K_1(C(S_H^5)).$$

Proof. Lemma 3.1 allows us to apply [R-J12, Corollary 1.5] to the family of surjections in diagram (2.3). The first six-term exact sequence is

$$\begin{array}{ccccc}
 K_0(P_1) & \longrightarrow & K_0(\mathcal{T}^{\otimes 2} \otimes C(S^1)) \oplus K_0(\mathcal{T} \otimes C(S^1) \otimes \mathcal{T}) & \dashrightarrow & K_0(\mathcal{T} \otimes C(S^1)^{\otimes 2}) \\
 \uparrow & & & & \downarrow \partial_{01} \\
 K_1(\mathcal{T} \otimes C(S^1)^{\otimes 2}) & \longleftarrow & K_1(\mathcal{T}^{\otimes 2} \otimes C(S^1)) \oplus K_1(\mathcal{T} \otimes C(S^1) \otimes \mathcal{T}) & \longleftarrow & K_1(P_1).
 \end{array}
 \tag{3.5}$$

Here the dotted arrows are $(\text{id} \otimes \sigma \otimes \text{id})_* - (\text{id} \otimes \text{id} \otimes \sigma)_*$. With the help of the Künneth formula the exact sequence becomes:

$$\begin{array}{ccccc}
 K_0(P_1) & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \dashrightarrow^{(m,n) \mapsto (m-n,0)} & \mathbb{Z} \oplus \mathbb{Z} \\
 \uparrow & & & & \downarrow \\
 \mathbb{Z} \oplus \mathbb{Z} & \longleftarrow^{(m,n) \mapsto (m,-n)} & \mathbb{Z} \oplus \mathbb{Z} & \longleftarrow & K_1(P_1).
 \end{array}
 \tag{3.6}$$

Hence $K_0(P_1) = \mathbb{Z} = K_1(P_1)$.

The second diagram of [R-J12, Corollary 1.5] is

$$\begin{array}{ccccc}
 K_0(P_2) & \longrightarrow & K_0(C(S^1) \otimes \mathcal{T} \otimes C(S^1)) \oplus K_0(C(S^1)^{\otimes 2} \otimes \mathcal{T}) & \dashrightarrow & K_0(C(S^1)^{\otimes 3}) \\
 \partial_{10} \uparrow & & & & \downarrow \partial_{01} \\
 K_1(C(S^1)^{\otimes 3}) & \longleftarrow & K_1(C(S^1) \otimes \mathcal{T} \otimes C(S^1)) \oplus K_1(C(S^1)^{\otimes 2} \otimes \mathcal{T}) & \longleftarrow & K_1(P_2).
 \end{array}
 \tag{3.7}$$

In order to unravel this diagram, we need to take a closer look into the Künneth formula. We consider $[u] \otimes [u] \in K_1(C(S^1)) \otimes K_1(C(S^1))$ and denote its image under the Künneth isomorphism $K_1(C(S^1)) \otimes K_1(C(S^1)) \rightarrow K_0(C(S^1) \otimes C(S^1))$ by β . Using the natural leg numbering convention, we extend this notation to triple tensor products with $[1] \in K_0(\mathcal{T})$ or with $[1] \in K_0(C(S^1))$ as an appropriate factor. Next, we denote by u_i the K_1 -class of a triple tensor with u as the i -th factor and $1 \in \mathcal{T}$ or $1 \in C(S^1)$ as any remaining factor.

Hence $K_1(C(S^1)^{\otimes 3})$ is \mathbb{Z}^4 generated by u_1, u_2, u_3 and the fourth generator denoted by u_{123} . Furthermore, the above exact sequence becomes

$$\begin{array}{ccccc} K_0(P_2) & \longrightarrow & \mathbb{Z}[1] \oplus \mathbb{Z}\beta_{13} \oplus \mathbb{Z}[1] \oplus \mathbb{Z}\beta_{12} & \cdots \cdots \cdots \rightarrow & \mathbb{Z}[1] \oplus \mathbb{Z}\beta_{12} \oplus \mathbb{Z}\beta_{13} \oplus \mathbb{Z}\beta_{23} \\ \uparrow & & & & \downarrow \\ \mathbb{Z}u_1 \oplus \mathbb{Z}u_2 \oplus \mathbb{Z}u_3 \oplus \mathbb{Z}u_{123} & \cdots \cdots \cdots \leftarrow & \mathbb{Z}u_1 \oplus \mathbb{Z}u_3 \oplus \mathbb{Z}u_1 \oplus \mathbb{Z}u_2 & \longleftarrow & K_1(P_2). \end{array}$$

Now, by the functoriality of the Künneth isomorphism [B-B98, p. 232] and with the help of the diagram (3.13), it is straightforward to verify that the upper and the lower dotted maps are respectively given by

$$(3.8) \quad (a, b, c, d) \mapsto (a - c, -d, b, 0) \quad \text{and} \quad (a, b, c, d) \mapsto (a - c, -d, b, 0).$$

Hence, by a straightforward homological computation, we infer that

$$(3.9) \quad K_0(P_2) = \mathbb{Z}[1] \oplus \mathbb{Z}\partial_{10}(u_{123}) \quad \text{and} \quad K_1(P_2) = \mathbb{Z}[\mathbf{u}_1, \mathbf{u}_1] \oplus \mathbb{Z}\partial_{01}(\beta_{23}),$$

where $\mathbf{u}_1 := u \otimes 1 \otimes 1$.

Finally, the last diagram of [R-J12, Corollary 1.5] is

$$(3.10) \quad \begin{array}{ccccc} K_0(C(S_H^5)) & \longrightarrow & K_0(P_1) \oplus K_0(C(S^1) \otimes \mathcal{T}^{\otimes 2}) & \cdots \cdots \cdots \rightarrow & K_0(P_2) \\ \uparrow & & & & \downarrow \\ K_1(P_2) & \cdots \cdots \cdots \leftarrow & K_1(P_1) \oplus K_1(C(S^1) \otimes \mathcal{T}^{\otimes 2}) & \longleftarrow & K_1(C(S_H^5)). \end{array}$$

Plugging in generators into this diagram, we obtain

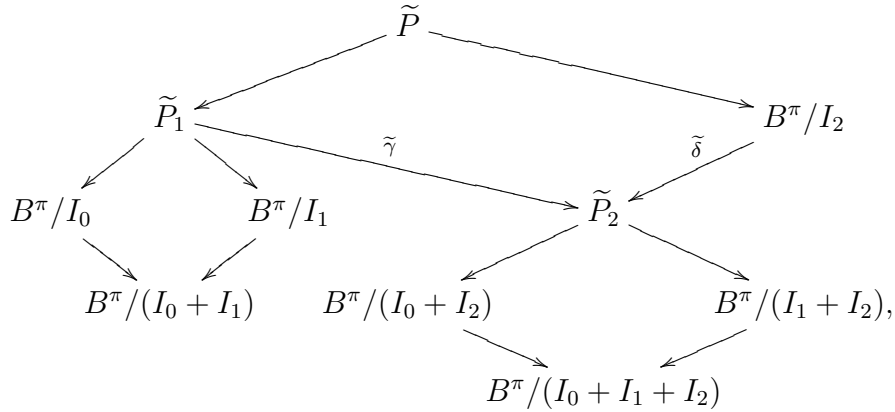
$$(3.11) \quad \begin{array}{ccccc} K_0(C(S_H^5)) & \longrightarrow & \mathbb{Z}[1] \oplus \mathbb{Z}[1] & \cdots \cdots \cdots \rightarrow & \mathbb{Z}[1] \oplus \mathbb{Z}\partial_{10}(u_{123}) \\ \uparrow & & & & \downarrow \\ \mathbb{Z}[\mathbf{u}_1, \mathbf{u}_1] \oplus \mathbb{Z}\partial_{01}(\beta_{23}) & \cdots \cdots \cdots \leftarrow & \mathbb{Z}\partial_{01}(\beta_{23}) \oplus \mathbb{Z}u_1 & \longleftarrow & K_1(C(S_H^5)). \end{array}$$

Here the upper dotted arrow is evidently given by the formula $(a, b) \mapsto (a - b, 0)$. It is a bit more complicated to determine the lower dotted arrow. To this end, we denote by $\mathbf{b} \in M_2(C(S^1) \otimes C(S^1))$ the pullback of the Bott projection on S^2 , so that $[\mathbf{b}] = \beta$. Next, by $\bar{\mathbf{b}} \in M_2(\mathcal{T} \otimes C(S^1))$ we denote a self-adjoint lifting of \mathbf{b} along $\text{id}_{M_2(\mathbb{C})} \otimes (\sigma \otimes \text{id})$. Then we substitute $\bar{\mathbf{b}}$ to the formula (1.13) to compute both $\partial_{01}(\beta_{23}) \in K_1(P_1)$ and $\partial_{01}(\beta_{23}) \in K_1(P_2)$ at the same time. The resulting formulas will only differ in the leftmost tensor factor: for P_1 it will be $1 \in \mathcal{T}$ and for P_2 it will be $1 \in C(S^1)$. Therefore

$$(3.12) \quad (\sigma_1, \sigma_1)_* : K_1(P_1) \ni \partial_{01}(\beta_{23}) \mapsto \partial_{01}(\beta_{23}) \in K_1(P_2).$$

Combining this observation with the diagram

(3.13)



one easily checks that the desired lower dotted map is given by the formula $(a, b) \mapsto (-b, a)$. Consequently, $K_0(C(S_H^5)) = \mathbb{Z} = K_0(C(S_H^5))$ as claimed. \square

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